Resonances in coupled-channel scattering from lattice QCD

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Based on work in collaboration with J.J. Dudek, R.G. Edwards and C.E. Thomas.

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Resonances from QCD

$$\Gamma(s) \sim \frac{1}{\rho(s) m_R^2} \frac{m_R^2 - s - is^{1/2} \Gamma(s)}{s^{1/2} \Gamma(s)}$$

$$m_R = 854.1 \pm 1.1 \text{ MeV}$$

$$g = 5.80 \pm 0.11$$

$$\Gamma_R = \frac{g^2 p_R^3}{6\pi m_R^2} = 12.4 \pm 0.6 \text{ MeV}$$

- $L = 1.9 \text{ fm}$
- $L = 2.4 \text{ fm}$
- $L = 2.9 \text{ fm}$

$m_\pi = 391 \text{ MeV}$
Coupled-channel scattering

• Most physical resonances couple to multiple channels.

• To understand the physical spectrum, applying coupled-channel methods will be essential.

• We consider here $\pi K$, where $\eta K$ can also contribute in $I=1/2$.

• The physical amplitudes have resonances in several partial waves

**Aim:** Obtain the scattering $S$-matrix from Lattice QCD
Resonances with Strangeness

Seen in Kaon beam experiments

\[ p \rightarrow K^- \rightarrow K^* \rightarrow \pi^+ K, \eta K, \ldots \]

LASS at SLAC \( E_K = 11 \) GeV.
Resonances with Strangeness

Seen in Kaon beam experiments

\( K^- \rightarrow K^* \rightarrow \pi^K, \eta K, \ldots \)

LASS at SLAC \( E_k = 11 \text{ GeV} \)
Resonances with Strangeness: Partial Waves

<table>
<thead>
<tr>
<th>$J^P$</th>
<th>States</th>
</tr>
</thead>
<tbody>
<tr>
<td>$0^+$</td>
<td>$\kappa$, $K_0^*(1430)$, ...</td>
</tr>
<tr>
<td>$1^-$</td>
<td>$K^*(892)$, ...</td>
</tr>
<tr>
<td>$2^+$</td>
<td>$K_2^*(1430)$, ...</td>
</tr>
</tbody>
</table>
Lattice calculation

- Large basis of operators including:
  
  "Single-meson" like operators, including bilinears and derivatives.

  "Meson-meson" like operators: Made from pairs of projected variationally-optimised single-meson operators at source and sink with definite momentum, e.g.:
  \[
  \Omega_\pi(\vec{p}_1)\Omega_K(\vec{p}_2)
  \]

- Include all Wick contractions.

- All relevant irreps and moving frames with
  \[
  p^2 = |\vec{p}_1 + \vec{p}_2|^2 \leq 4 \left( \frac{2\pi}{L} \right)^2
  \]

\[
C_{ij}(t) = \langle 0 | \mathcal{O}_i^\dagger(t) \mathcal{O}_j(0) | 0 \rangle
\]

\[
\mathcal{O}_i = \bar{\psi} \Gamma \overleftrightarrow{D} \ldots \overleftrightarrow{D} \psi
\]

\[
C(t)\nu^n(t) = \lambda_n(t)C(t_0)\nu^n(t)
\]

\[
\Omega_n^\dagger = \sum_i \nu^n_i \mathcal{O}_i^\dagger
\]
Coupled-channel scattering

\[ a_t E_{cm} \]

- \( A_1^+ \)
- \( \eta' K \) \text{thr.}
- \( \eta[011]K[011] \)
- \( \pi[011]K[011] \)
- \( \eta[001]K[001] \)
- \( \pi[001]K[001] \)
- \( \eta K \) \text{thr.}
- \( \eta[000]K[000] \)
- \( \pi K \) \text{thr.}
- \( \pi[000]K[000] \)

\[ L/\alpha_s \]

0.16  0.20  0.24  0.28

16  20  24
Coupled-channel scattering from lattice QCD
Coupled-channel scattering from lattice QCD
Coupled-channel extensions of Lüscher’s method

\[ \text{det} \left[ t_{ij}^{-1}(E) + \mathcal{M}_{ij}(E) \right] = 0 \]

Many contributors:
Lüscher
Gottlieb & Rummukainen
Christ, Kim, & Yamazaki
Kim, Sachrajda & Sharpe
He, Feng & Liu
Bernard, Lage, Meissner, and Rusetsky
Leskovec & Prelovsek
Briceño & Davoudi
Hansen & Sharpe
Gockeler et al
Guo, Dudek, Edwards & Szczepaniak
Briceño, Davoudi, Luu
+ ...
Coupled-channel extensions of Lüscher’s method

\[
\det \left[ t_{ij}^{-1}(E) + \mathcal{M}_{ij}(E) \right] = 0
\]

\[ S = 1 + 2i\rho \, t \]

diagonal in partial waves, mixes channels

infinite volume scattering \( t \)-matrix

known finite-volume functions
diagonal in channels, mixes partial waves
Coupled-channel extensions of Lüscher’s method

$$\det \left[ t_{ij}^{-1}(E) + M_{ij}(E) \right] = 0$$
Coupled-channel scattering

Problem: Three or more unknowns for each energy level, eg:

\[ S_{11} = \eta e^{2i\delta_{\pi K}} \]

\[ S_{22} = \eta e^{2i\delta_{\eta K}} \]

2x2 complex matrix (or more) but only one equation.
No one-to-one relation from energy levels to amplitudes

Finite volume energy levels

Scattering Amplitudes

Extensions of Lüscher’s method
**Coupled-channel scattering**

**Solution:** Parameterise $t$-matrix, constrain parameters using many energy levels

**E.g.:** $K$-matrix (it’s essential that we preserve unitarity)

\[
S_{ij} = \delta_{ij} + 2i \left( \rho_i \rho_j \right)^{\frac{1}{2}} t_{ij}
\]

\[
[S^\dagger S]_{ij} = \delta_{ij}
\]

\[
\rightarrow \text{Im}[t^{-1}]_{ij} = -\rho_i \delta_{ij}
\]

- $K$-matrix contains everything that isn’t constrained by unitarity

\[
t_{ij}^{-1}(s) = K_{ij}^{-1}(s) - i\delta_{ij} \rho_i(s)
\]

- $K$ must be real for real $s$. One option for two channel scattering:

\[
K = \frac{1}{m^2 - s} \begin{bmatrix}
    g_{\pi K}^2 & g_{\pi K} g_{\eta K} \\
    g_{\pi K} g_{\eta K} & g_{\eta K}^2
\end{bmatrix} + \begin{bmatrix}
    \gamma_{\pi K, \pi K} & \gamma_{\pi K, \eta K} \\
    \gamma_{\pi K, \eta K} & \gamma_{\eta K, \eta K}
\end{bmatrix}
\]

- $m, g, \gamma$ are real free parameters. Simple to add more - more poles, or a polynomial in $s$.

- Simple to generalise to scattering with non-zero angular momentum.

- Can improve model by adding extra physically motivated properties - eg: Chew-Mandelstam phase space.
Coupled-channel scattering

- Describe $t$-matrix using $K$-matrix in $S$-wave only $\rightarrow$ obtain a spectrum.
- Minimise a $\chi^2$ to obtain the best agreement between the $K$-matrix and lattice energies.

$$K = \frac{1}{m^2 - s} \begin{bmatrix} g_{\pi K}^2 & g_{\pi K} g_{\eta K} & g_{\eta K}^2 \end{bmatrix} + \begin{bmatrix} \gamma_{\pi K, \pi K} & \gamma_{\pi K, \eta K} \\ \gamma_{\pi K, \eta K} & \gamma_{\eta K, \eta K} \end{bmatrix}.$$
Coupled-channel scattering

- Describe $t$-matrix using $K$-matrix in $S$-wave only $\rightarrow$ obtain a spectrum.
- Minimise a $\chi^2$ to obtain the best agreement between the $K$-matrix and lattice energies.

\[ K = \frac{1}{m^2 - s} \begin{bmatrix} g_{\pi K}^2 & g_{\pi K} g_{\eta K} \\ g_{\pi K} g_{\eta K} & g_{\eta K}^2 \end{bmatrix} + \begin{bmatrix} \gamma_{\pi K, \pi K} & \gamma_{\pi K, \eta K} \\ \gamma_{\pi K, \eta K} & \gamma_{\eta K, \eta K} \end{bmatrix}. \]

\[ \chi^2/N_{\text{dof}} = \frac{6.40}{15 - 6} = 0.71 \]
$S$-wave amplitudes

- Broad resonance in $S$-wave $\pi K$.
- $\eta K$ coupling is small.
- 3 subthreshold points, naturally included in an energy-level fit.

$$S_{11} = \eta e^{2i\delta_{\pi K}}$$

$$S_{22} = \eta e^{2i\delta_{\eta K}}$$

$\chi^2/N_{\text{dof}} = \frac{6.40}{15-6} = 0.71$. 

\begin{align*}
m &= (0.2466 \pm 0.0020 \pm 0.0009) \cdot a_t^{-1} \\
g_{\pi K} &= (0.165 \pm 0.006 \pm 0.002) \cdot a_t^{-1} \\
g_{\eta K} &= (0.033 \pm 0.010 \pm 0.003) \cdot a_t^{-1} \\
\gamma_{\pi K, \pi K} &= 0.184 \pm 0.054 \pm 0.030 \\
\gamma_{\pi K, \eta K} &= -0.52 \pm 0.20 \pm 0.06 \\
\gamma_{\eta K, \eta K} &= -0.37 \pm 0.07 \pm 0.05
\end{align*}
More energy levels

- Many more energy levels from irreps where the mesons are **moving with respect to the lattice**.
- More than 100 usable levels.
$S+P$-waves from 80 energy levels

\[ \chi^2 / N_{\text{dof}} = \frac{49.1}{61-6} = 0.89 \]

\[ \chi^2 / N_{\text{dof}} = \frac{15.0}{19-4} = 1.00 \]

- Separate fits and global fits yield consistent results.
- $D$-wave is negligible in this region.
Parameterisation variation

- Separate fits and global fits yield consistent results.
- $D$-wave is negligible in this region.
Narrow $D$-wave resonance

- Many other energy levels containing scattering amplitude information.

- Using only irreps with $\tilde{J}=2$ and higher ($E^+, T_2^+, [100]B_{1,2}$) we find a narrow resonance:

- Fit to energies.

- In $\tilde{J} \geq 1$ scattering the lowest threshold is $\pi\pi K$ at $a_t E_{cm}=0.235$.

- Ideally requires 3-body formalism. Although not strictly rigorous, we can apply the $2 \rightarrow 2$ formalism anyway.
$S$-matrix poles

\[ m = \text{Re} \sqrt{s_0} / \text{MeV} \]

\[ \Gamma = 2 \cdot \text{Im} \sqrt{s_0} / \text{MeV} \]
S-matrix poles

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Coupled-channel scattering from lattice QCD
$S$-matrix poles

$m = \text{Re} \sqrt{s_0} / \text{MeV}$

$\Gamma = 2 \cdot \text{Im} \sqrt{s_0} / \text{MeV}$

Coupled-channel scattering from lattice QCD
S-matrix poles

\[ m = \text{Re}\sqrt{s_0} / \text{MeV} \]

\[ \Gamma = 2 \cdot \text{Im}\sqrt{s_0} / \text{MeV} \]

\[ \delta_2 \]

\[ \chi^2/N_{\text{dof}} = 0.89 \]
**S-matrix S-wave poles**

\[ t_{ij}(s \sim s_0) \sim \frac{c_i c_j}{S_0 - s} \]

- **K-matrix pole + const**
- **K-matrix pole + linear**
  - \( K^{-1} \) poly \{1,0,1\}
  - \( K^{-1} \) poly \{2,0,1\}
  - \( K^{-1} \) poly \{1,1,1\}
  - \( K^{-1} \) poly \{1,0,0\}
  - \( K^{-1} \) poly \{2,0,0\}
  - \( K^{-1} \) poly \{2,1,0\}

- **elastic scat. len.**
- **elastic eff. range.**

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**Coupled-channel scattering from lattice QCD**
$S$-wave amplitudes vs experiment

(a) $|a_s|$ vs $M_{K^-\pi^+}$ (GeV/c$^2$)

(b) Phase $\phi_s$ vs $M_{K^-\pi^+}$ (GeV/c$^2$)
Summary

- Coupled-channel scattering amplitudes can be obtained from QCD using lattice methods.
- Using extensions of Lüscher’s method, we were able to connect finite volume energy levels to infinite volume scattering amplitudes.
- There are many exciting possibilities for future calculations using similar methods:
  - Strongly coupled systems like the $a_0(980)$ and $f_0(980)$ are under investigation.
  - Investigations into $\pi\gamma \rightarrow \pi\pi$ and similar processes are underway.
  - Channels involving charm quarks are also under investigation by European collaborators.
- Further in the future: $\pi N \rightarrow \pi N, \gamma N \rightarrow \pi N$. Multiparticle scattering, exotics.
Coming soon: $\pi\eta$-$K\bar{K}$-$\pi\eta'$

\begin{align*}
L/\alpha_s &
\begin{array}{cccc}
16 & 18 & 20 & 22 & 24 \\
0.16 & 0.18 & 0.20 & 0.22 & 0.24 & 0.26 & 0.28 & 0.30
\end{array}
\end{align*}
Backup slides: Lattice
$P$-wave contributions

Coupled-channel scattering from lattice QCD
• Overlaps $\sim$ guide to resonant content
  \[ Z^n_i = \langle n | O^\dagger_i | 0 \rangle \]

• Shifted $\pi K$-like and $\eta K$-like states

• $\mathcal{J}^P=1^-$ state near to $\pi K$ threshold, $\mathcal{J}^P=2^+$ state, extra $\mathcal{J}^P=0^+$.

• Considerable partial-wave mixing.

[011] $A_1$ contains $\mathcal{J}^P=0^+$, 1, 2, ...

$\sim J^P = 2^+$

$\sim J^P = 0^+$

$\sim J^P = 1^-$
P-wave near-threshold state

Elastic scattering just above $\pi K$ threshold, no $\eta K$ to consider.

The irreps with $P$-wave overlap:


all have an “extra” level near $\pi K$ threshold.

Fitting the energy levels using an elastic Breit-Wigner in $\pi K$: 

\[ a_t E_{\text{cm}} \]
**P-wave near-threshold state**

Elastic scattering just above $\pi K$ threshold, no $\eta K$ to consider.

The irreps with $P$-wave overlap:


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Fitting the energy levels using an elastic Breit-Wigner in $\pi K$:

\[
\Gamma(s) = \frac{g_R^2 k_{cm}^3}{6\pi E_{cm}^2}
\]

\[
t = \frac{1}{\rho(s) m_R^2 - s - is^{3/2} \Gamma(s)}
\]

\[
k^3 \cot \delta_1 = (m_R^2 - s) \frac{6\pi s^{1/2}}{g_R^2}
\]

In $t$ there is a pole on the real axis just below $\pi K$ threshold:

Bound state in $J^P=1^-$. 

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Coupled-channel scattering from lattice QCD
Coupled-channel calculation details

• Large basis of operators including:

  “Single-meson” like operators, including bilinears and derivatives.

  \[ \mathcal{O}_i = \bar{\psi} \Gamma \hat{D} \ldots \hat{D} \psi \]

  “Meson-meson” like operators: Made from pairs of projected variationally-optimised single-meson operators at source and sink with definite momentum, e.g.:

  \[ \Omega_\pi(\tilde{p}_1) \Omega_K(\tilde{p}_2) \]

  \[ C(t) \nu^n(t) = \lambda_n(t) C(t_0) \nu^n(t) \]

  \[ \Omega_n^\dagger = \sum_i \nu_i^n \mathcal{O}_i^\dagger \]

• Include all Wick contractions.

• All relevant irreps with boosts

  \[ p^2 = |\tilde{p}_1 + \tilde{p}_2|^2 \leq 4 \left( \frac{2\pi}{L} \right)^2 \]
Relative operator overlaps

\[ Z_i = \langle n | O_i | 0 \rangle \]
Excited state spectra from lattice QCD

Operators with overall momentum

Because momentum is quantised, different energies can be accessed by considering operators with an overall momentum.

\[ \vec{p} = \frac{2\pi}{\xi L} \vec{n} \]

\[ E_{\text{lat}}^2 = E_{\text{cm}}^2 + \left( \frac{2\pi}{\xi L} |\vec{n}| \right)^2 \]

Useful to consider systems with \( \vec{n} = (0,0,0), (1,0,0), (1,1,0), (1,1,1), (2,0,0) \)

Overall zero momentum: \( \pi(0,0,0)\pi(0,0,0), \pi(1,0,0)\pi(-1,0,0), \ldots \)

One unit: \( \pi(1,0,0)\pi(0,0,0), \pi(1,1,0)\pi(-1,0,0), \ldots \)

Less symmetry: More mixing of angular momentum!
Principal correlators

\[ \lambda_0 \]
\[ \chi^2 / N_{\text{dof}} = 0.76 \]
\[ a_t E_{\text{lat}} = 0.16541(66) \]

\[ \lambda_3 \]
\[ \chi^2 / N_{\text{dof}} = 0.68 \]
\[ a_t E_{\text{lat}} = 0.26802(99) \]

\[ \lambda_6 \]
\[ \chi^2 / N_{\text{dof}} = 0.62 \]
\[ a_t E_{\text{lat}} = 0.3373(54) \]

\[ \lambda_1 \]
\[ \chi^2 / N_{\text{dof}} = 0.38 \]
\[ a_t E_{\text{lat}} = 0.17793(66) \]

\[ \lambda_4 \]
\[ \chi^2 / N_{\text{dof}} = 0.72 \]
\[ a_t E_{\text{lat}} = 0.27666(84) \]

\[ \lambda_7 \]
\[ \chi^2 / N_{\text{dof}} = 1.02 \]
\[ a_t E_{\text{lat}} = 0.3400(113) \]

\[ \lambda_2 \]
\[ \chi^2 / N_{\text{dof}} = 0.58 \]
\[ a_t E_{\text{lat}} = 0.23097(68) \]

\[ \lambda_5 \]
\[ \chi^2 / N_{\text{dof}} = 1.00 \]
\[ a_t E_{\text{lat}} = 0.3364(29) \]
The starting point is the path integral:

\[
\langle x_b | e^{-iHT} | x_a \rangle = \int \mathcal{D}x(t) e^{iS[x(t)]}
\]
Path integrals

To solve numerically, consider the discretised version

\[ \langle x_b | e^{-iHT} | x_a \rangle = \prod_{x_{t_i}} \int dx_{t_i} e^{iS(x_{t_i})} \]
Path integrals

Evaluate correlation functions from the path integral:

\[
\langle 0 | \mathcal{O}_i(t) \mathcal{O}_j^\dagger(0) | 0 \rangle = \frac{1}{Z_0} \int \mathcal{D} \bar{\psi} \mathcal{D} \psi \mathcal{D} A \ \mathcal{O}_i(t) \mathcal{O}_j^\dagger(0) \ e^{iS[\bar{\psi}, \psi, A]}
\]

In order to deal with strong coupling: Solve the QCD path integral numerically.

Integrate over gauge field configurations: Infinitely many possibilities.

➔ Store field values on a discrete set of points
Lattice QCD

Use a finite spacetime volume, $L^3 \times t$. ($L$ Roughly 2-3fm in these studies).
Use a finite number of points, with separation $a \sim 0.1$fm. ($L/a = 16, 20, 24$)

Quarks live on discrete points and the gluons live on the links between them.

Use periodic boundary conditions: Volume becomes a torus.
Use a finite spacetime volume, $L^3 \times t$. ($L$ Roughly 2-3fm in these studies).
Use a finite number of points, with separation $a \sim 0.1$fm ($L/a = 16, 20, 24$).
Change variables to Euclidean spacetime to simplify integration.

Quarks live on discrete points and the gluons live on the links between them.

Use periodic boundary conditions: Volume becomes a torus.
Correlation functions

Evaluate correlation functions from the path integral:

\[ C_{ij}(t) = \langle 0 | \mathcal{O}_i(t) \mathcal{O}_j^\dagger(0) | 0 \rangle \]
\[ = \frac{1}{Z_0} \int \mathcal{D} \bar{\psi} \mathcal{D} \psi \mathcal{D} A \ \mathcal{O}_i(t) \mathcal{O}_j^\dagger(0) \ e^{-S[\bar{\psi}, \psi, A]} \]

Leads to the ground state energy for large t:

\[ C_{ij}(t) = \sum_n \frac{1}{2E_n} \langle 0 | \mathcal{O}_i | n \rangle \langle n | \mathcal{O}_j^\dagger | 0 \rangle \ e^{-E_n t} \]
\[ = \frac{Z_i^* Z_j}{2E_n} e^{-E_n t} \]

The symmetries of the operators dictate which states can be extracted

\[ \bar{\psi} \Gamma \psi \]
\[ \begin{align*}
\Gamma & \quad \gamma_5 \sim \pi \quad J^P = 0^- \\
\gamma_i \sim \rho \quad J^P = 1^- 
\end{align*} \]
Extracting a spectrum

Getting the ground state is useful, but we want to extract the whole spectrum in a finite volume.

Fitting subleading exponentials doesn’t get very far:
With very precise data, sometimes a second state can be found.

A solution: The variational method.

\[ C_{ij}(t)v^n_j = \lambda_n(t)C_{ij}(t_0)v^n_j \]

If more than one operator overlaps onto the same state represented by some eigenvector \( v^n_i \) the generalised eigenvalue problem can be solved and then as many states as operators may be extracted.

\[ \lambda_n(t) \sim e^{-E_n(t-t_0)} \]

... a large basis of operators are needed
Operators and the variational method

\[ C_{ij}(t)v_j^n = \lambda_n(t)C_{ij}(t_0)v_j^n \]
\[ \lambda_n(t) \sim e^{-E_n(t-t_0)} \]

Use a large basis of operators

\[ \Theta_i = \bar{\psi} \Gamma \psi \]
\[ \Theta_i = \bar{\psi} \Gamma \mathring{D} \ldots \mathring{D} \psi \]

\[ \Gamma_i = \{1, \gamma_0, \gamma_5, \gamma_0\gamma_5, \gamma_i, \gamma_0\gamma_i, \gamma_5\gamma_i, [\gamma_i, \gamma_j] \} \]

Use the variational method with a large correlation matrix
Symmetry on the lattice

The lattice has a cubic symmetry.
It does not have the O(3) symmetry of continuous space.

Eg: 2D QM

Continuous rotational spatial symmetry

$$e^{i\phi} \rightarrow e^{i\phi + i \alpha}$$

$$e^{i\phi} \rightarrow e^{i\phi + in\pi/2}$$

Only symmetric at discrete angles
Symmetry on the lattice

Continuous rotational spatial symmetry

\[ e^{i\phi} \rightarrow e^{i\phi + i\alpha} \quad \text{vs} \quad e^{i\phi} \rightarrow e^{i\phi + in\pi/2} \]

Only symmetric at discrete angles

Cubic symmetry groups mix the continuum angular momentum:

<table>
<thead>
<tr>
<th>Irrep</th>
<th>( J^p )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( A_1^+ )</td>
<td>0(^+), 4(^+), ...</td>
</tr>
<tr>
<td>( T_1^- )</td>
<td>1(^-), 3(^-), ...</td>
</tr>
</tbody>
</table>
Backup slides: Finite volume formalism
Two particles in a finite volume

Simple 1-d problem

No interactions $\rightarrow$ total energy is just the sum

\[ E = \left( p_1^2 + m_1^2 \right)^{\frac{1}{2}} + \left( p_2^2 + m_2^2 \right)^{\frac{1}{2}} \]

For a single particle:

\[ p_i^2 = \left( \frac{2\pi n}{L} \right)^2 \]

Non-interacting energies in a finite volume are known from the single-particle analysis

If we measure the energies on the lattice and find a difference, this shift must be due to interactions.

Lüscher et al
Two particles in a finite volume

In simple QM: Interactions lead to phase shift $\delta$ on the wavefunction $\psi(x) \sim e^{\pm ipx}$

Periodic boundary conditions for interacting particles.

$\psi(0) = \psi(L), \quad \frac{\partial \psi}{\partial x} \bigg|_{x=0} = \frac{\partial \psi}{\partial x} \bigg|_{x=L}$

$\sin \left( \frac{pL}{2} + \delta(p) \right) = 0$

$p = \frac{2\pi n}{L} - 2 \frac{L}{L} \delta(p)$

Discrete spectrum of allowed energies directly connected to the phase shift.

If we measure the energies on the lattice and find a difference, this shift must be due to interactions.
Two particles in a finite volume

In 3+1 dimensions, this leads to a simple relation between the finite volume energy and the S-wave scattering length:

\[ k \cot \delta = \frac{1}{a} + \frac{1}{2} r k^2 + O(k^4) \]

\[ = \frac{1}{\pi L} \sum_{\vec{n} \in \mathbb{Z}_3} \frac{1}{\left( |\vec{n}|^2 - \left( |\vec{k}|L/(2\pi) \right)^2 \right)^2} \]
Finite volume spectra

Weak interactions

Small, +ve scattering length (weakly attractive)
Weakly repulsive scattering from QCD

\( \pi K \rightarrow \pi K, \ I = 3/2 \)
Resonances

Narrow resonance:
- Pole in $t$, close to real axis.

\[ t \sim \frac{1}{\rho(s)} \frac{s^{\frac{1}{2}} \Gamma(s)}{m_R^2 - s - is^{\frac{1}{2}} \Gamma(s)} \]

Broad resonance:
- Pole in $t$, far from real axis

\[ \Gamma(s) = \frac{g_R^2}{6\pi} \frac{k_{cm}}{s} \]
Finite volume spectra with a resonance

Weak interactions

Coupled-channel scattering from lattice QCD
Finite volume spectra with a resonance

Narrow resonance
Finite volume spectra with a resonance

Narrow resonance
Finite volume spectra with a resonance

Weak interactions

Coupled-channel scattering from lattice QCD
Finite volume spectra with a resonance

- Broad resonance

Coupled-channel scattering from lattice QCD
Finite volume spectra with a resonance

Extra level due to the resonance ~ within the resonance width.
To understand the state better we can study it in several volumes and apply Lüscher’s method and extensions.
Finite volume spectra with a resonance

Broad resonance

using $L/a = 16^3, 20^3, 24^3$
Finite volume spectra with a resonance

Many energy levels can map out the scattering amplitude
Coupled-channel extensions of Lüscher’s method

\[
S_{ij} = \delta_{ij} + 2i \left( \rho_i \rho_j \right)^{\frac{1}{2}} t_{ij}
\]

\[
\det \left[ \delta_{ij} \delta_{\ell\ell'} \delta_{nn'} + i \rho_i t_{ij} \right] \left( \delta_{\ell\ell'} \delta_{nn'} + \hat{M}_{ij, \ell n, \ell' n'} \right) = 0
\]

Channels: eg \( \pi K, \eta K \)

Angular momentum

scattering t-matrix

momentum boost vector

lattice irrep

finite volume object - contains generalised Lüscher Zeta functions

\[
M \sim \frac{1}{\gamma} \sum_{\text{spins}} (\text{CGs}) \sum_{\hat{\mathbf{r}}} \frac{r^\ell Y_{\ell m}(\hat{\mathbf{r}})}{r^2 - q^2}
\]

Symmetry of the volume mixes partial waves - \( M \) mixes partial waves.

\( t \)-matrix is diagonal in partial waves, but can couple scattering channels: \( \pi K \to \eta K \)
Coupled-channel scattering

\[ \mathcal{M} \sim \frac{1}{\gamma} \sum_{\text{spins}} (\text{CGs}) \sum_{\mathbf{r}} \frac{r^{\ell} Y_{\ell m} (\mathbf{r})}{r^2 - q^2} \]

scattering t-matrix, couples channels, diagonal in \( l \).

\[
\det \left[ \delta_{ij} \delta_{\ell \ell'} \delta_{nn'} + i \rho_i t_{ij}^{(\ell)} \left( \delta_{\ell \ell'} \delta_{nn'} + i \mathcal{M}_{ij, \ell n, \ell' n'} \right) \right] = 0
\]

finite volume object - contains generalised Lüscher Zeta functions mixes partial waves

- Several unknowns at each energy level: Multiple channels, multiple partial waves.
- Problem is unconstrained for a single energy level.
- Solution: Parameterise \( t_{ij} \) using a few free parameters, use many energy levels to constrain them.
Example Minimisation

\[ \delta_0^{\pi K}, \delta_0^{\eta K}, \eta, \chi^2 \]

\[ L/\alpha_s \]

\[ m, g_{\pi K}, \gamma_{\eta K, \eta K} \]

credit: Jo Dudek
Backup slides: Amplitudes
P-wave pole

- Breit-Wigner pole from continuation below threshold.
- Also used a K-matrix below threshold, found almost exactly the same result.
- Poles on physical sheet $\text{Im}(k_{\text{cm}}) > 0$.
**S-wave poles**

Multi-sheeted complex plane due to square-root branch cuts at each threshold, in single channel case for now:

\[ k_{cm} = \pm \frac{1}{2} \left( E_{cm}^2 - 4m^2 \right)^{\frac{1}{2}} \]

---

**Bound state**

**Resonance**

**Virtual Bound state**
$S$-wave poles

Multi-sheeted complex plane due to square-root branch cuts at each threshold, in single channel case for now:

$$k_{cm} = \pm \frac{1}{2} \left( E_{cm}^2 - 4m^2 \right)^{\frac{1}{2}}$$

**Bound state**

**Resonance**

**Virtual Bound state**
Poles

Multi-sheeted complex plane due to square-root branch cuts at each threshold, in single channel case for now:

\[ k_{cm} = \pm \frac{1}{2} \left( E_{cm}^2 - 4m^2 \right)^{\frac{1}{2}} \]

Bound state
Poles

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**Bound state**

**Resonance**
Poles

Multi-sheeted complex plane due to square-root branch cuts at each threshold, in single channel case for now:

\[ k_{\text{cm}} = \pm \frac{1}{2} \left( E_{\text{cm}}^2 - 4m^2 \right)^{\frac{1}{2}} \]
**S-wave poles**

Actual situation: Unequal masses and an extra pair of sheets due to $\eta K$ scattering

\[ \kappa_{cm} = \pm \frac{1}{2} \left( E_{cm}^2 - 2 \left( m_1^2 + m_2^2 \right) + \left( \frac{m_1^2 - m_2^2}{E_{cm}} \right)^2 \right)^{\frac{1}{2}} \]
Virtual bound state $\kappa$

Pelaez and Nebreda using Unitarised SU(3) Chiral Perturbation theory
More on virtual bound state

In an effective range parameterisation, strong interactions near threshold lead to a large $a$.

In $S$-wave large $a$ automatically leads to a pole near-threshold.

\[ k_{\text{cm}} \cot \delta_0 = \frac{1}{a} + \frac{1}{2} r k_{\text{cm}}^2 \]

\[ t = \frac{1}{2} \frac{E_{\text{cm}}}{k_{\text{cm}} - \frac{i}{a}} \quad k_{\text{cm}} = \mp \frac{i}{a} \]

Arguments appear to hold for constant terms in K-matrix (slightly complicated by Chew-Mandelstam).

Appears to break down for $P$-wave and higher.

![Graph showing $|k_{\text{cm}}|$ vs $\delta_0$ and $a, E_{\text{cm}}$ vs $\delta_0$](image)
More \( K \)-matrix details

- \( K \)-matrix contains everything not constrained by unitarity

\[
t_{ij}^{-1}(s) = K^{-1}_{ij}(s) - i\delta_{ij}\rho_i(s)
\]

\[
K = \frac{1}{m^2 - s} \left[ \begin{array}{cc} g_{\pi\pi}^2 & g_{\pi\eta} \, g_{\eta\eta} \\ g_{\pi\eta} \, g_{\eta\eta} & g_{\eta\eta}^2 \end{array} \right] + \left[ \begin{array}{cc} \gamma_{\pi\pi,\pi\pi} & \gamma_{\pi\pi,\eta\eta} \\ \gamma_{\pi\eta,\pi\pi} & \gamma_{\eta\eta,\eta\eta} \end{array} \right]
\]

- Chew-Mandelstam phase space -- include also \( s \)-channel cut along with imaginary part.

\[
t_{ij}^{-1}(s) = K^{-1}_{ij}(s) + \delta_{ij} \, I_i(s)
\]

\[
I_i(s) = I_i(s_{thr}) - \frac{s - s_{thr}}{\pi} \int_{s_{thr}}^{\infty} ds' \frac{\rho_i(s')}{(s' - s)(s' - s_{thr})}
\]

(Substring at pole so that \( \text{Re} \, I(s = m^2) = 0 \))

- Threshold factors for \( l>0 \)

\[
t_{ij}^{-1}(s) = \frac{1}{(2k_i)^{\ell}} \, K^{-1}_{ij}(s) \frac{1}{(2k_j)^{\ell}} + \delta_{ij} \, I_i(s)
\]

As used in Guo, Mitchell and Szczepaniak Phys.Rev. D82 (2010) 094002

No modifications were used in \( I(s) \) for higher waves.

Also tested phase space factors instead of \( k_i \) for thresholds.
Virtual bound state $\kappa$

Pelaez and Nebreda using Unitarised SU(3) Chiral Perturbation theory
Backup slides: $\rho$ resonance
Extracting the $\rho$ resonance

Several volumes: $L=16$, 20, 24.
Operators in several moving frames, upto $n=(2,0,0)$.

Anisotropic lattices:
temporal spacing 3.5 times finer for better energy resolution.

Combination of single particle and meson-meson operators.

$m_\pi=391$ MeV
Finite volume spectra in I=1 J=1

$\vec{p} = [000] T_1$

$\vec{p} = [011] A_1$

$\vec{p} = [111] A_1$

$\vec{p} = [001] A_1$

$\vec{p} = [011] B_1$

$\vec{p} = [111] E_2$

$\vec{p} = [001] E_2$

$\vec{p} = [011] B_2$

$\vec{p} = [002] A_1$
A resonance from QCD

\[ \Gamma(s) = \frac{g_R^2}{6\pi} \frac{k_{cm}^3}{E_{cm}^2} \]

\[ \pi \rightarrow \pi \rightarrow 1 \longrightarrow \pi \longrightarrow (s) \]

\[ m_R = 854.1 \pm 1.1 \text{ MeV} \]
\[ g = 5.80 \pm 0.11 \]

\[ \Gamma_R = \frac{g^2}{6\pi} \frac{p_R^2}{m_R^2} = 12.4 \pm 0.6 \text{ MeV} \]

- \( L = 1.9 \text{ fm} \)
- \( L = 2.4 \text{ fm} \)
- \( L = 2.9 \text{ fm} \)

\( m_\pi = 391 \text{ MeV} \)
Strong coupling

Lagrangian of QCD

\[ \mathcal{L}_{\text{QCD}} = \sum_q \bar{\psi}_q (i\mathcal{D} - m_q) \psi_q - \frac{1}{4} \mathcal{F}_{\mu\nu} \mathcal{F}^{\mu\nu} \]

\[ \mathcal{D} = \gamma^\mu (\partial_\mu - igA_\mu) \]

\[ \mathcal{F}_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + g [A_\mu, A_\nu] \]

Coloured quark and gluon degrees of freedom.

- Excited state spectrum contains many interesting open questions.
- Interesting effects near thresholds: tetraquarks? meson-meson bound states? \( f_0(980) \) in \( \pi\pi \) scattering and new charmonium states, eg: \( Z(4430) \).
- Hybrid states, containing explicit gluonic degrees of freedom. Could be seen in new experiments, like GlueX.
Strong coupling

Lagrangian of QCD

$$\mathcal{L}_{\text{QCD}} = \sum_q \bar{\psi}_q \left( i\slashed{D} - m_q \right) \psi_q - \frac{1}{4} F_{\mu\nu} F^{\mu\nu}$$

$$\slashed{D} = \gamma^\mu \left( \partial_\mu - ig A_\mu \right)$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + g [A_\mu, A_\nu]$$

Many interesting consequences:
Confinement - no asymptotic quarks or gluons.
Dynamical chiral symmetry breaking.
Light physical pion ~ goldstone boson of the symmetry breaking.
Cannot use perturbation theory: Non-perturbative methods needed.

Several options including: Models
Schwinger-Dyson+Bethe-Salpeter
Effective Field Theories
Lattice QCD
Coupled channel scattering

**Calculation details:**

Use lattice QCD to obtain finite volume energy levels and use the Lüscher method and its extensions to connect to infinite volume physics.

\[ m_\pi = 391 \text{ MeV}. \]

Three different lattices:
16\(^3\), 20\(^3\), 24\(^3\) spatial lattice sites.
Corresponds to boxes with sides 1.9, 2.4, 2.9 fm.

Several boosts in each volume - gives more energy levels (in finite volume). \[ \vec{p}_i^2 = \left( \frac{2\pi n}{L} \right)^2 \]

Use the cubic group irreducible representations to extract partial wave information.