Precise determination of $\pi N$ scattering and the $\sigma_{\pi N}$

J. Ruiz de Elvira

Helmholtz-Institut für Strahlen- und Kernphysik (Theorie) and Bethe Center for Theoretical Physics, Universität Bonn

In collaboration with:
M. Hoferichter, B. Kubis, U.-G. Meißner.

PWA 8 / ATHOS 3, Ashburn, April 16, 2015

In memory of G. Höhler (passed away on June 4, 2014)
Motivation: Why $\pi N$ scattering?

- **Low energies**: test chiral dynamics in the baryon sector
  $\Rightarrow$ low-energy theorems e.g. for the scattering lengths

- **Higher energies**: resonances, baryon spectrum

- **Input for $NN$ scattering**: LECs $c_i$, $\pi NN$ coupling

- **Crossed channel $\pi \pi \rightarrow \bar{N}N$**: nucleon form factors
  $\Rightarrow$ probe the structure of the nucleon
  - spectral functions of form factors
  - vector form factors (P-waves)
  - scalar form factors (S-waves)
The pion-nucleon $\sigma$-term

Scalar form factor of the nucleon:

$$\sigma(t) = \langle N(p') | \hat{m} (\bar{u}u + \bar{d}d) | N(p) \rangle \quad t = (p' - p)^2 \quad \sigma_{\pi N} = \sigma(0)$$

- $\sigma_{\pi N}$ measures the light-quark contribution to the nucleon mass
- Unfortunately, no direct experimental access to it
- Linked to $\pi N$ via the Cheng-Dashen theorem

\[F_\pi^2 \bar{D}^+(\nu = 0, t = 2M^2_\pi) = \sigma(2M^2_\pi) + \Delta_R\]

\[F_\pi^2 (d_{00}^+ + 2M^2_\pi d_{01}^+) + \Delta_D = \sigma_{\pi N} + \Delta_\sigma\]

$$|\Delta_R| \lesssim 2 \text{ MeV} \quad [\text{Bernard, Kaiser, Meißner 1996}]$$

$$\Delta_D - \Delta_\sigma = (-1.8 \pm 0.2) \text{ MeV} \quad [\text{Gasser, Leutwyler, Sainio 1991}]$$
Phenomenological status

- **Karlsruhe/Helsinki** partial-wave analysis KH80 Höhler et al. 1980s
  ← comprehensive analyticity constraints, old data

- Formalism for the extraction of $\sigma_{\pi N}$ via the **Cheng–Dashen low-energy theorem**
  Gasser, Leutwyler, Locher, Sainio 1988, Gasser, Leutwyler, Sainio 1991
  ← “canonical value” $\sigma_{\pi N} \sim 45$ MeV, based on KH80 input

- **GWU/SAID** partial-wave analysis Pavan, Strakovsky, Workman, Arndt 2002
  ← much larger value $\sigma_{\pi N} = (64 \pm 8)$ MeV

- More recently: ChPT in different regularizations (w/ and w/o $\Delta$) Alarcón et al. 2012
  ← fit to PWAs, $\sigma_{\pi N} = 59 \pm 7$ MeV
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- This talk: two new sources of information on low-energy $\pi N$ scattering
  - Precision extraction of $\pi N$ **scattering lengths** from **hadronic atoms**
    - [Baru et al. 2011]
  - **Roy-equation** constraints: analyticity, unitarity, crossing symmetry
Motivation: Why Roy-Steiner equations?

Roy(-Steiner) eqs. = Partial-Wave (Hyberbolic) Dispersion Relations coupled by unitarity and crossing symmetry

- **Respect all symmetries**: analyticity, unitarity, crossing
- **Model independent** ⇒ the actual parametrization of the data is irrelevant once it is used in the integral.
- Framework allows for **systematic improvements** (subtractions, higher partial waves, ...)
- **PW(H)DRs** help to study processes with **high precision**:
  - \(\pi\pi\)-scattering: [Ananthanarayan et al. (2001), García-Martín et al. (2011)]
  - \(\pi K\)-scattering: [Büttiker et al. (2004)]
  - \(\gamma\gamma \rightarrow \pi\pi\) scattering: [Hoferichter et al. (2011)]
Warm up: Roy-equations for $\pi\pi$

- $\pi\pi \rightarrow \pi\pi \Rightarrow$ fully crossing symmetric in Mandelstam variables $s$, $t$, and $u = 4M_\pi - s - t$

- Start from twice-subtracted fixed-$t$ DRs of the generic form

$$T^l(s, t) = c(t) + \frac{1}{\pi} \int_{4m_\pi^2}^\infty \frac{ds'}{s'^2} \left[ \frac{s^2}{(s' - s)} - \frac{u^2}{(s' - u)} \right] \text{Im}T^l(s', t)$$

- Subtraction functions $c(t)$ are determined via crossing symmetry functions of the $I=0,2$ scattering lengths: $a_0^0$ and $a_2^0$

- PW-expansion of these DRs yields the **Roy-equations** [Roy (1971)]

$$t_J^l(s) = ST^l_J(s) + \sum_{J'} (2J' + 1) \sum_{l'=0,1,2} \int_{4m_\pi^2}^\infty ds' K^l_{JJ'}(s', s) \text{Im} t_J^{l'}(s')$$

- $K^l_{JJ'}(s', s) \equiv$ kernels $\Rightarrow$ analytically known
Solving Roy-equations: flow information

- **Roy-equations** rigorously valid for a finite energy range ⇒ introduce a matching point $s_m$
- only partial waves with $J \leq J_{\text{max}}$ are solved
- assume isospin limit

**Input**
- High-energy region: $\text{Im} t_{IJ}(s)$ for $s \geq s_m$ and for all $J$
- Higher partial waves: $\text{Im} t_{IJ}(s)$ for $J > J_{\text{max}}$ and for all $s$

**Output**
- Self-consistent solution for $\delta_{IJ}(s)$ for $J \leq J_{\text{max}}$ and $s_{th} \leq s \leq s_m$
- Constraints on subtraction constants
Roy-Steiner equations for $\pi N$: difficulties

Key difficulties compared to $\pi \pi$ Roy-equations

- **Crossing**: coupling between $\pi N \to \pi N$ (s-channel) and $\pi \pi \to \bar{N}N$ (t-channel)

  \[ \Rightarrow \text{hyperbolic dispersion relations} \quad \text{[Hite, Steiner 1973], [Büttiker et al. 2004]} \]

- **Unitarity** in t-channel, e.g. in single-channel approximation

\[ \text{Im} f^J_{\pm}(t) = \sigma_{t}^{\pi} f^J_{\pm}(t) t^J_{J}(t)^* \]

  \[ \Rightarrow \text{Watson’s theorem: phase of } f^J_{\pm}(t) \text{ equals } \delta_{IJ} \quad \text{[Watson 1954]} \]

solve with Muskhelishvili-Omnès techniques

\[ \Rightarrow \text{Omnès function: } \Omega^J_{J}(t) = \exp \left\{ \frac{t}{\pi} \int_{t_{th}}^{t_{m}} dt' \frac{\delta^J_{J}(t)}{t'(t'-t)} \right\} \]

- **Large pseudo-physical region in t-channel**

  \[ \Rightarrow \bar{K}K \text{ intermediate states for s-wave in the region of the } f_0(980) \]
Solving t-channel: single channel

- Elastic-channel approximation: generic form of the integral equation

\[
f(t) = \Delta(t) + (a + bt)(t - 4m^2) + \frac{t^2(t - 4m^2)}{\pi} \int_{i\pi}^{\infty} \frac{\text{Im} f(t')}{t'(t'^2 - 4m^2)(t' - t)} \mathrm{d}t'
\]

- \( \Delta(t) \): Born terms, s-channel integrals, higher t-channel partial waves
  \( \Rightarrow \) left-hand cut
- Introduce subtractions at \( \nu = t = 0 \) \( \Rightarrow \) subthreshold parameters \( a, b \)
- Solution in terms of Omnès function:

\[
f(t) = \Delta(t) + (t - 4m^2)\Omega(t)(1 - t\dot{\Omega}(0))a + t(t - 4m^2)\Omega(t)b
\]

\[
- \Omega(t) \frac{t^2(t - 4m^2)}{\pi} \left\{ \int_{4M^2/\pi}^{t_{\text{m}}} \frac{\Delta(t')\text{Im} \Omega(t')^{-1}}{t'^2(t' - 4m^2)(t' - t)} \mathrm{d}t' + \int_{t_{\text{m}}}^{\infty} \frac{\Omega(t')^{-1}\text{Im} f(t')}{t'(t' - 4m^2)(t' - t)} \mathrm{d}t' \right\}
\]

\[
\Omega(t) = \exp \left\{ \frac{t}{\pi} \int_{i\pi}^{t_{\text{m}}} \frac{\mathrm{d}t'}{t'} \frac{\delta(t')}{t' - t} \right\}
\]

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Solving t-channel: input and subtractions

- elastic channel approximation: $\sqrt{t_m} = 0.98 - 1.1$ GeV, for $t > t_m \text{ Im} f^J_{\pm}(t) = 0$
- First step: check consistency with KH80 [Höhler 1983]
- Input needed:
  - $\pi\pi$ phase shifts: [Caprini, Colangelo, Leutwyler, (in preparation)], [Madrid group]
  - $\pi N$ phase shifts: SAID [Arndt et al. 2008], KH80
  - $\pi N$ at high energies: Regge model [Huang et al. 2010]
  - $\pi N$ parameters: KH80
Solving t-channel: P, D and F waves up to $\bar{N}N$
Solving t-channel: S-wave results

- Important for the $\pi N$ term
- $\bar{K}K$ channel important $\Rightarrow$ two-channel Muskhelishvili-Omnès problem
- Information on $\pi K$ and $KN$ is also needed:

MO solutions in general consistent with KH80 results
Solving s-channel: flow of information

- General form of the s-channel integral equation

\[ f_{l+}^I(W) = \Delta_{l+}^I(W) + \frac{1}{\pi} \int_{W_+}^{\infty} dW' \sum_{l'=0}^{\infty} \left\{ K_{ll'}^I(W, W') \text{Im} f_{l'}^I(W') + K_{ll'}^I(W, -W') \text{Im} f_{(l'+1)-}^I(W') \right\} \]

\[ \leftrightarrow \text{form of } \pi\pi \text{ Roy-Equations} \]

- \( \Delta_{l+}^I(W) \equiv \text{subtraction constants, } t\text{-channel contribution and pole term} \)
- valid up to \( W_m = 1.38 \text{ GeV} \)

**Input:**

- RS t-channel solutions
- s-channel partial waves for \( J > 1 \) [SAID, KH80]
- s-channel partial waves for \( W_m < W < 2.5 \text{ GeV} \) [SAID, KH80]
- high energy contribution for \( W > 2.5 \text{ GeV} \): Regge model [Huang et al. 2010]

**Output:**

- Self-consistent solution for S and P waves between \( s_{\text{th}} \leq s \leq s_m \)
- Constraints on subtraction constants \( \Rightarrow \) subthreshold parameters
Solving s-channel: strategy

- Parametrize S and P waves up to $W < W_m$
  - Using SAID partial waves as starting point
- Impose as a constraint scattering lengths from a combined analysis of pionic hydrogen and deuterium [Baru et al. 2011]

$$a_0^{1/2} = (169.8 \pm 2.0) \times 10^{-3} M_\pi^{-1} \quad a_0^{3/2} = (-86.3 \pm 1.8) \times 10^{-3} M_\pi^{-1}$$

- Introduce as many subtractions as necessary to match d.o.f [Gasser, Wanders 1999]
- Minimize difference between LHS and the RHS on a grid of points $W_j$

$$\chi^2 = \sum_{l,I_s,\pm} \sum_{j=1}^N \left( \frac{\text{Re} f^I_{l\pm}(W_j) - F[f^I_{l\pm}](W_j)}{\text{Re} f^I_{l\pm}(W_j)} \right)^2$$

$F[f^I_{l\pm}](W_j) \equiv$ right hand side of RS-equations

- Parametrization and subthreshold parameters are the fitting parameters
Solving $s$-channel: results

- $s_{11}(s)$
- $s_{31}(s)$
- $p_{13}(s)$
- $p_{33}(s)$
- $p_{11}(s)$
- $p_{31}(s)$

Notation: $L_{2I_s 2J}$

Blue/red

LHS/RHS

after the fit

Gray/black

LHS/RHS

before the fit

Precise $\pi N$ determination
Solving the full RS system: strategy

- Full solution: self-consistent, **iterative** solution of the **full RS** system
  ⇒ consistent set of **s-** and **t-channel** PWs & **low-energy parameters**

- However:
  - **t-channel RS** eqs. depend only weakly on **s-channel** PWs
  - resulting **s-channel** PW change little from **SAID**

A **full solution** can be achieved including in the **s-channel** RS eqs. the **t-channel** dependence on the **subthreshold parameters**
Solving the full RS system: uncertainties

- **Statistical errors** *(at intermediate energies)*
  - important correlations between subthreshold parameters
  - shallow fit minima
  - Sum rules for subthreshold parameters become essential to reduce the errors

- **Input variation** *(small)*
  - small effect for considering s-channel KH80 input
  - very small effects from $L > 5$ s-channel PWs
  - small effect from the different S-wave extrapolation for $t > 1.3$ GeV
  - negligible effect of $\rho'$ and $\rho''$
  - very significant effects of the D-waves ($f_2(1275)$)
  - F-waves shown to be negligible

- **Matching conditions** *(close to $W_m$)*
- **Scattering length (SL) errors** *(on S-waves and subthreshold parameters)*
  - very important for the $\sigma_{\pi N}$
Uncertainties: s-channel pw
Uncertainties: t-channel pw

Motivation
Introduction
RS-eqs.
Summary
Comparison with KH80

- **Karlsruhe-Helsinki analysis KH80** [Höhler et al. 1980]
  - comprehensive *analyticity constraints* based on fixed-t dispersion relations
  - old experimental data

- Here, an update of **KH80** results with modern input
  - HDR increase the **range of validity** of the equations
  - $\pi N$ scattering length extracted from hadronic atoms $\Rightarrow$ essential for the $\sigma_{\pi N}$
  - Goldberger-Miyazawa-Oehme sum rule:

    $g_{\pi N}^2/4\pi = 13.7 \pm 0.2$ [Baru et al. 2011]

    compare:

    $g_{\pi N}^2/4\pi = 14.28$ [Höhler et al. 1983]

- s-channel PWs from **SAID**
- $f_2(1275)$ included $\Rightarrow$ sizable effect

- **KH80** is *internally consistent* $\Rightarrow$ RS reproduces **KH80** results with **KH80** input
Results: s-channel PWs with KH80 input

Motivation
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Notation: $L_{2I_s 2J}$

Precise $\pi N$ determination
Results: t-channel PWs with KH80 input

before the fit

after the fit

KH80
Solving the full RS system: $\sigma_{\pi N}$

**Results for the $\sigma_{\pi N}$**

\[
\sigma_{\pi N} = \Sigma_d + \Delta_D - \Delta_R - \Delta_\sigma
\]

\[
\Sigma_d = F_\pi^2 \left( d_{00}^+ + 2M_\pi^2 d_{01}^+ \right), \quad \Delta_D - \Delta_R - \Delta_\sigma = -(1.8 \pm 2.2) \text{ MeV}
\]

\[
\Sigma_d = 57.9 \pm 1.8 \text{ MeV}
\]

\[
\sigma_{\pi N} = (56.1 \pm 2.8) \text{ MeV} + (3 \pm 2.2) \text{ MeV (IV)}
\]

⇒ with KH80 SL+(KH80 input): $\Sigma_d = 47.9 \text{ MeV} \Rightarrow \sigma_{\pi N} = 46 \text{ MeV}$

to be compared with $\sigma_{\pi N} = 45$ [Gasser, Leutwyler, Socher, Sainio 1988, Gasser, Leutwyler, Sainio 1991]

⇒ KH80 is internally **consistent** but at odd with the modern SL determinations
What has been done:

- Derived a closed system of Roy-Steiner equations (PWHDRs) for $\pi N$ scattering
- Constructed unitarity relations including $\bar{K}K$ intermediate states for the t-channel PWs
- Implemented subtractions at several orders
- Solved the t-channel MO problem for a single- and two-channel approximation
  $\Rightarrow$ t-channel RS/MO machinery works
- Numerical solution of the full system of RS eqs.
- Precise determination of the $\sigma_{\pi N}$
- Error analysis

What needs to be done:

- Extraction of the Low Energy Constants
- Possible improvements: higher PWs, more inelastic input, ...
Spare slides
Roy-Steiner equations for $\pi N$: flow of information

Higher partial waves
$\text{Im } f_{l \pm}^l, l \geq 2, s \leq s_m$

$s$-channel partial waves
solve Roy–Steiner equations for $s \leq s_m$

Inelasticities
$\eta_{l \pm}^l, l \leq 1, s \leq s_m$

Subtraction constants
$\pi N$ coupling constant

$t$-channel partial waves
solve Roy–Steiner equations for $t \leq t_m$

$\pi\pi$ scattering phases $\delta_J^l$

High-energy region
$\text{Im } f_{l \pm}^l, s \geq s_m$

High-energy region
$\text{Im } f_{l \pm}^l, t \geq t_m$
Solving Roy-Steiner equations for $\pi N$: Recoupling schemes

**s-channel** subproblem:

- Kernels are diagonal for $I \in \{+, -\}$, but unitarity relations are diagonal for $I_s \in \{1/2, 3/2\} \Rightarrow$ all partial-waves are interrelated
- Once the t-channel PWs are known
  \[ \Rightarrow \] Structure similar to $\pi \pi$ Roy-equations

**t-channel** subproblem:

- Only higher PWs couple to lower ones
- Only PWs with even or odd J are coupled
- No contribution from $f_J^+$ to $f_{J+1}^-$
  \[ \Rightarrow \] Leads to Muskhelishvili-Omnès problem
\( \pi^a(q) + N(p) \rightarrow \pi^b(q') + N(p') \)

- **Isospin Structure:**
  \[ T^{ba} = \delta^{ba} T^+ + \epsilon^{ab} T^- \]

- **Lorentz Structure:** \( I \in \{+,-\} \)
  \[ T^I = \bar{u}(p') \left( A^I + \frac{q + q'}{2} B^I \right) u(p) \]
  \[ D^I = A^I + \nu B^I, \quad \nu = \frac{s-u}{4m} \]

- **Isospin basis:** \( I_s \in \{1/2, 3/2\} \)
  \( \{T^+, T^-\} \Leftrightarrow T^{1/2}, T^{3/2} \)

- **PW projection:**
  - s-channel pw: \( f^I_{\pm} \)
  - t-channel pw: \( f^J_{\pm} \)
  **Bose symmetry** \( \Rightarrow \) even/odd \( J \Leftrightarrow I = +/− \)
\(\pi N\)-scattering basics: partial waves

- **s-channel** projection:

  \[
  f_{l\pm}^I(W) = \frac{1}{16\pi W} \left\{ (E + m) [A_l^I(s) + (W - m) B_l^I(s)] + (E - m) [-A_{l\pm 1}^I(s) + (W + m) B_{l\pm 1}^I(s)] \right\}
  \]

  \[
  X_l^I(s) = \int_{-1}^{1} dz_s \, P_l(z_s) X(s,t) \bigg|_{t=t(s,z_s)=-2q^2(1-z_s)} \quad \text{for } X \in \{A, B\} \text{ and } W = \sqrt{s}
  \]

- **McDowell symmetry**:

  \[
  f_{l+}^I(W) = -f_{(l+1)-}(-W) \quad \forall \ l \geq 0
  \]

- **t-channel** projection:

  \[
  f_+^J(t) = -\frac{1}{4\pi} \int_0^1 dz_t \, P_J(z_t) \left\{ \frac{p_t^2}{(p_t q_t)^J} A_l^I(s,t) \bigg|_{s=s(t,z_t)} - \frac{m}{(p_t q_t)^{J-1}} z_t B_l^I(s,t) \bigg|_{s=s(t,z_t)} \right\} \quad \forall \ J \geq 0
  \]

  \[
  f_-^J(t) = \frac{1}{4\pi} \frac{\sqrt{J(J+1)}}{2J+1} \frac{1}{(p_t q_t)^{J-1}} \int_0^1 dz_t \left[ P_{J-1}(z_t) - P_{J+1}(z_t) \right] B_l^I(s,t) \bigg|_{s=s(t,z_t)} \quad \forall \ J \geq 1
  \]

- **Bose symmetry** \(\Rightarrow\) even/odd \(J \iff I = +/−\)
Roy-Steiner equations for $\pi N$: HDR’s

- **Hyperbolic DRs:**
  
  $$(s - a)(u - a) = b = (s' - a)(u' - a) \text{ with } a, b \in \mathbb{R}$$

  $$A^+(s, t; a) = \frac{1}{\pi} \int_{s^+}^{\infty} ds' \left[ \frac{1}{s' - s} + \frac{1}{s' - u} - \frac{1}{s' - a} \right] \text{Im} A^+(s', t') + \frac{1}{\pi} \int_{t_{\pi}}^{\infty} dt' \frac{\text{Im} A^+(s', t')}{t' - t}$$

- **Why HDR?**
  - Combine all physical regions $\Rightarrow$ crucial for t-channel projection
  - Evade double-spectral regions $\Rightarrow$ the PW decompositions converge
  - No kinematical cuts, manageable kernel functions

- **Similar derivation to $\pi \pi$ Roy equations.**
  - Expand imaginary parts in terms of s- and t-channel partial waves
  - Project onto s- and t-channel partial waves
  - Combine the resulting equations using s- and t-channel PW unitarity relations

- **Validity:**
  
  $W \in [W_+ = 1.08 \text{ GeV}, 1.38 \text{ GeV}], \sqrt{t} \in [\sqrt{t_{\pi}} = 0.28 \text{ GeV}, 2.00 \text{ GeV}]$.

- **Subtractions:** subthreshold expansion around $\nu = t = 0$

  $$\overline{A}^+(\nu, t) = \sum_{m,n=0}^{\infty} a_{mn}^+ \nu^{2m} t^n$$
Roy-Steiner equations for $\pi N$: HDR’s

**Hyperbolic DRs:** $(s - a)(u - a) = b = (s' - a)(u' - a)$ with $a, b \in \mathbb{R}$

$$A^+(s, t; a) = \frac{1}{\pi} \int_{s_+}^{\infty} ds' \left[ \frac{1}{s' - s} + \frac{1}{s' - u} - \frac{1}{s' - a} \right] \text{Im} A^+(s', t') + \frac{1}{\pi} \int_{t_\pi}^{\infty} dt' \frac{\text{Im} A^+(s', t')}{t' - t}$$

$$B^+(s, t; a) = N^+(s, t) + \frac{1}{\pi} \int_{s_+}^{\infty} ds' \left[ \frac{1}{s' - s} - \frac{1}{s' - u} \right] \text{Im} B^+(s', t') + \frac{1}{\pi} \int_{t_\pi}^{\infty} dt' \frac{\nu \text{Im} B^+(s', t')}{\nu' t' - t}$$

$$N^+(s, t) = g^2 \left( \frac{1}{m^2 - s} - \frac{1}{m^2 - u} \right)$$

similar for $A^-, B^-$ and $N^-$ [Hite/Steiner (1973)]

**Why HDR?**

- Combine all physical regions ⇒ crucial for t-channel projection
- Evade double-spectral regions ⇒ the PW decompositions converge
- Range of convergence can be maximized by tuning the free hyperbola parameter $a$
- No kinematical cuts, manageable kernel functions
πN-scattering basics: Unitarity relations

- **s-channel** unitarity relations \((I_s \in \{1/2, 3/2\})\):

\[
\text{Im} f^{I_s}_{i \pm}(W) = q |f^{I_s}_{i \pm}(W)|^2 \theta(W - W_+) + \frac{1 - (\eta^{I_s}_{i \pm}(W))^2}{4q} \theta(W - W_{\text{inel}})
\]

- **t-channel** unitarity relations: 2-body intermediate states: \(\pi \pi + \bar{K}K + \cdots\)

\[
\text{Im} f^J_{\pm}(t) = \sigma^\pi_I (t^J_{f I}(t))^* f^J_{\pm}(t) \theta(t - t_\pi) + 2c_J \sqrt{2} k^J_k \sigma^K_I (g^J_J(t))^* h^J_{\pm}(t) \theta(t - t_K)
\]

- Only linear in \(f^J_{\pm}(t) \Rightarrow \text{less restrictive}\)
Recipe to derive **Roy-Steiner** equations:
- Expand imaginary parts in terms of s- and t-channel partial waves
- Project onto s- and t-channel partial waves
- Combine the resulting equations using s- and t-channel PW unitarity relations

Similar structure to $\pi\pi$ **Roy equations**

**Validity**: assuming Mandelstam analyticity

- s-channel ⇒ optimal for $a = -23.2M^2_\pi$
  
  $$s \in [s_+ = (m + M_\pi)^2, 97.30M^2_\pi] \iff W \in [W_+ = 1.08 \text{ GeV}, 1.38 \text{ GeV}]$$

- t-channel ⇒ optimal for $a = -2.71M^2_\pi$
  
  $$t \in [t_\pi = 4M^2_\pi, 205.45M^2_\pi] \iff \sqrt{t} \in [\sqrt{t_\pi} = 0.28 \text{ GeV}, 2.00 \text{ GeV}]$$
Roy-Steiner equations for $\pi N$: subtractions

- **Subtractions** are necessary to ensure the convergence of DR integrals
  $\Rightarrow$ asymptotic behavior
- Can be introduced to lessen the dependence of the low-energy solution on the high-energy behavior
- Parametrize high-energy information in (a priori unknown) subtraction constants
  $\Rightarrow$ matching to ChPT
- Subthreshold expansion around $\nu = t = 0$

\[
\begin{align*}
\bar{A}^+ (\nu, t) &= \sum_{m,n=0}^{\infty} a^+_{mn} \nu^{2m} t^n \\
\bar{B}^+ (\nu, t) &= \sum_{m,n=0}^{\infty} b^+_{mn} \nu^{2m+1} t^n \\
\bar{A}^- (\nu, t) &= \sum_{m,n=0}^{\infty} a^-_{mn} \nu^{2m+1} t^n \\
\bar{B}^- (\nu, t) &= \sum_{m,n=0}^{\infty} b^-_{mn} \nu^{2m} t^n,
\end{align*}
\]

where

\[
\begin{align*}
\bar{A}^+ (s, t) &= A^+ (s, t) - \frac{g^2}{m} \\
\bar{B}^+ (s, t) &= B^+ (s, t) - g^2 \left[ \frac{1}{m^2 - s} - \frac{1}{m^2 - u} \right] \\
\bar{A}^- (s, t) &= A^- (s, t) \\
\bar{B}^- (s, t) &= B^- (s, t) - g^2 \left[ \frac{1}{m^2 - s} + \frac{1}{m^2 - u} \right] + \frac{g^2}{2m^2},
\end{align*}
\]
RS-eqs for $\pi N$: Range of convergence

- Subthreshold expansion around $\nu = t = 0$

  \[
  A^+ (\nu, t) = \frac{g^2}{m} + d^{+}_{00} + d^{+}_{01} t + a^{+}_{10} \nu^2 + \mathcal{O} (\nu^2, t^2) \\
  A^- (\nu, t) = \nu a^{+}_{00} + a^{+}_{01} \nu t + a^-_{10} \nu^3 + \mathcal{O} (\nu^5, \nu t^2, \nu^3 t) \\
  B^+ (\nu, t) = g^2 \frac{4 m \nu}{(m^2 - s_0)^2} + \nu b^{+}_{00} + \mathcal{O} (\nu^3, \nu t) \\
  B^- (\nu, t) = g^2 \left[ \frac{2}{m^2 - s_0} - \frac{t}{(m^2 - s_0)^2} \right] - \frac{g^2}{2 m^2} + b^-_{00} + b^-_{01} t + b^-_{10} \nu^2 + \mathcal{O} (\nu^2, \nu^2 t, t^2)
  \]

- Pseudovector Born terms: $D^l = A^l + \nu B^l$

  \[
  \bar{D}^+ = d^{+}_{00} + d^{+}_{01} t + d^{+}_{10} \nu^2 \\
  d^+_m = a^+_m + b^+_{m-1, n}, \quad d^-_m = a^-_m + b^-_n
  \]

- Sum rules for subthreshold parameters:

  \[
  d^{+}_{00} = - \frac{g^2}{m} + \frac{1}{\pi} \int_{s_+}^{\infty} ds' h_0 (s') \left[ \text{Im} A^+ (s', z_{s'}) \right]_{(0,0)} + \frac{1}{\pi} \int_{t_\pi}^{\infty} dt' \left[ \text{Im} A^+ (t', z_{t'}) \right]_{(0,0)} \\
  h_0 (s') = \frac{2}{s' - s_0} - \frac{1}{s' - a}
  \]
Roy-Steiner equations for $\pi N$: s-channel

s-channel RS equations

$$f^{I}_{l+}(W) = N^{I}_{l+}(W) + \frac{1}{\pi} \int_{W_{+}}^{\infty} dW' \sum_{l' = 0}^{\infty} \left\{ K^{I}_{ll'}(W, W') \text{Im} f^{I}_{l'}(W') + K^{I}_{ll'}(W, -W') \text{Im} f^{I}_{(l'+1)-}(W') \right\}$$

$$+ \frac{1}{\pi} \int_{t_{\pi}}^{\infty} dt' \sum_{J} \left\{ G^{J}_{ll'}(W, t') \text{Im} f^{J}_{+}(t') + H^{J}_{ll'}(W, t') \text{Im} f^{J}_{-}(t') \right\}$$

$$= -f^{I}_{(l+1)-}(W) \quad \forall \ l \geq 0 \ , \quad \text{[Hite/Steiner (1973)]}$$

- $K^{I}_{ll'}(W, W'), G^{J}_{ll'}(W, t')$ and $H^{J}_{ll'}(W, t')$-Kernels: analytically known,
  
  e.g. $K^{I}_{ll'}(W, W') = \frac{\delta^{ll'}_{W'}}{W'_{-}W} + \ldots \quad \forall \ l, l' \geq 0 \ ,$

- **Validity**: assuming Mandelstam analyticity
  
  $\Rightarrow$ optimal for $a = -23.2 M_{\pi}^2$

$$s \in \left[ s_{+} = (m + M_{\pi})^2, 97.30 \ M_{\pi}^2 \right] \ \Leftrightarrow \ W \in [W_{+} = 1.08 \ \text{GeV}, 1.38 \ \text{GeV}]$$
Roy-Steiner equations for $\pi N$: t-channel

**t-channel RS equations**

\[
\begin{align*}
    f^J_\pm(t) &= \tilde{N}^J_\pm(t) + \frac{1}{\pi} \int_{W+}^\infty dW' \sum_{l=0}^{\infty} \left\{ \tilde{G}^J_{Jl}(t, W') \Im f^J_{l+}(W') + \tilde{G}^J_{Jl}(t, -W') \Im f^J_{(l+1)-}(W') \right\} \\
    &\quad + \frac{1}{\pi} \int_{t_\pi}^{\infty} dt' \sum_{J'} \left\{ \tilde{K}^1_{JJ'}(t, t') \Im f^{J'}_{J+}(t') + \tilde{K}^2_{JJ'}(t, t') \Im f^{J'}_{J-}(t') \right\} \quad \forall J \geq 0 ,
    \\
    f^J_-(t) &= \tilde{N}^J_-(t) + \frac{1}{\pi} \int_{W+}^\infty dW' \sum_{l=0}^{\infty} \left\{ \tilde{H}^J_{Jl}(t, W') \Im f^J_{l+}(W') + \tilde{H}^J_{Jl}(t, -W') \Im f^J_{(l+1)-}(W') \right\} \\
    &\quad + \frac{1}{\pi} \int_{t_\pi}^{\infty} dt' \sum_{J'} \tilde{K}^3_{JJ'}(t, t') \Im f^{J'}_{J-}(t') \quad \forall J \geq 1 ,
\end{align*}
\]

**Validity:** assuming Mandelstam analyticity

$\Rightarrow$ optimal for $a = -2.71M^2_\pi$

\[
t \in \left[t_\pi = 4M^2_\pi, 205.45 M^2_\pi \right] \iff \sqrt{t} \in \left[\sqrt{t_\pi} = 0.28 \text{ GeV}, 2.00 \text{ GeV}\right].
\]
Solving t-channel: P-wave results

MO solutions in general consistent with KH80 results
Solving t-channel: coupled channels

- Generic coupled-channel integral equation

\[
f(t) = \Delta(t) + \frac{1}{\pi} \int_{t_{\pi}}^{t_{m}} \frac{d't'}{t' - t} \left( T^*(t') \Sigma(t') f(t') \right) + \frac{1}{\pi} \int_{t_{m}}^{\infty} \frac{d't'}{t' - t} \text{Im} f(t')
\]

- Formal solution as in the single-channel case (now with Omnès matrix \( \Omega(t) \))

\[
f(t) = \begin{pmatrix} f_0^+(t) \\ h_0^+(t) \end{pmatrix} \quad \text{Im} \Omega(t) = (T(t))^* \Sigma(t) \Omega(t)
\]

- Two linearly independent solutions \( \Omega_1, \Omega_2 \) [Muskhelishvili 1953]

- In general no analytical solution for the Omnès matrix but for its determinant [Moussallam 2000]

\[
det \Omega(t) = \exp \left\{ \frac{t}{\pi} \int_{t_{\pi}}^{t_{m}} d't' \frac{\psi(t')}{t'(t' - t)} \right\}.
\]
Solving t-channel S-wave equations: input

- Input needed:
  - $\pi\pi$ s-wave partial waves: [Caprini, Colangelo, Leutwyler, (in preparation)]
  - $K\bar{K}$ s-wave partial waves: [Büttiker. (2004)]
  - $\pi N$ and $KN$ s-wave pw: SAID [Arndt et al. 2008], KH80
  - $\pi N$ at high energies: Regge model [Huang et al. 2010]
  - $\pi N$ parameters: KH80
  - Hyperon couplings from [Jülich model 1989]
  - KN subthreshold parameters neglected

- Two-channel approximation breaks down at $\sqrt{t_0} = 1.3$ GeV $\Rightarrow 4\pi$ channel
- From $t_0$ to $t = 2$ GeV, different approximations considered
MO solutions in general consistent with KH80 results
Consistency with KH80

- parametrize SAID S and P waves up to $W < W_m$
  
  Imposing a continuous and differentiable matching point

- Compare between the input (LHS) and the output (RHS)

⇒ important discrepancies
Solving s-channel: consistency with KH80. P-waves
Solving s-channel: subtractions

- **Existence and uniqueness of solutions** [Gasser, Wanders 1999]
  ⇒ no-cusp condition for each pw + 2 additional constraints are needed

- Take advantage of the precise data for pionic atoms [Gotta et al. 2005, 2010]
  ⇒ Impose as a **constraint** scattering lengths from a combined analysis of pionic hydrogen and deuterium [Baru et al. 2011]

\[
a_{0+}^{1/2} = (169.8 \pm 2.0) \times 10^{-3} M_{\pi}^{-1} \quad a_{0+}^{3/2} = (-86.3 \pm 1.8) \times 10^{-3} M_{\pi}^{-1}
\]

\[
\text{Re} f_{l \pm}^I(s) = q^2 l \left( a_{l \pm}^I + b_{l \pm}^I q^2 + \cdots \right)
\]

**10 subthreshold parameters** are needed to match **d.o.f**

⇒ **three subtractions**
Results: s-channel PWs

\begin{align*}
&\text{blue/red} \\
&\LHS/\RHS \\
&\text{after the fit} \\
&\text{gray/black} \\
&\LHS/\RHS \\
&\text{before the fit} \\
&\text{Notation: } L_{2I_s2J}
\end{align*}
Results: s-channel PWs

**Motivation**
Introduction
RS-eqs.
Summary

**Results:**

- $s_{11}(s)$
- $s_{31}(s)$
- $p_{13}(s)$
- $p_{33}(s)$
- $p_{11}(s)$
- $p_{31}(s)$

**Notation:** $L_{2s_{2j}}$

Sizable $f_2(1275)$ effect

Blue/red

LHS/RHS

After the fit

Gray/black

LHS/RHS

Before the fit

Precise $\pi N$ determination

J. Ruiz de Elvira
Results: t-channel PWs

Motivation
Introduction
RS-eqs.
Summary

Results: t-channel PWs

blue

⇧

before the fit

red

⇧

after the fit

+ 

KH80

J. Ruiz de Elvira

Precise $\pi N$ determination

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### Full Solution: subthreshold parameters

<table>
<thead>
<tr>
<th></th>
<th>RS</th>
<th>KH80</th>
</tr>
</thead>
<tbody>
<tr>
<td>$d_{00}^+$</td>
<td>-1.36(3)</td>
<td>-1.46(10)</td>
</tr>
<tr>
<td>$b_{00}^-$</td>
<td>10.49(11)</td>
<td>10.36(10)</td>
</tr>
<tr>
<td>$a_{00}^-$</td>
<td>-9.08(11)</td>
<td>-8.83(10)</td>
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<tr>
<td>$d_{01}^+$</td>
<td>1.15(2)</td>
<td>1.14(2)</td>
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<tr>
<td>$b_{00}^+$</td>
<td>-3.45(7)</td>
<td>-3.54(6)</td>
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<tr>
<td>$a_{10}^+$</td>
<td>4.61(8)</td>
<td>4.66</td>
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<td>$b_{01}^-$</td>
<td>0.22(2)</td>
<td>0.24(1)</td>
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<tr>
<td>$a_{01}^-$</td>
<td>-0.35(2)</td>
<td>-0.37(2)</td>
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<tr>
<td>$b_{10}^-$</td>
<td>1.00(2)</td>
<td>1.08(5)</td>
</tr>
<tr>
<td>$a_{10}^-$</td>
<td>-1.16(3)</td>
<td>-1.25(5)</td>
</tr>
</tbody>
</table>
RS-eqs for $\pi N$: Range of convergence

- **Assumption**: Mandelstam analyticity [Mandelstam (1958, 1959)]

  $\Rightarrow T(s, t) \text{ can be written in terms double spectral densities: } \rho_{st}, \rho_{su}, \rho_{ut}$

  \[
  T(s, t) = \frac{1}{\pi^2} \int \int ds' du' \frac{\rho_{su}(s', u')}{(s' - s)(u' - u)} + \frac{1}{\pi^2} \int \int dt' du' \frac{\rho_{tu}(t', u')}{(t' - t)(u' - u)} + \frac{1}{\pi^2} \int \int ds' dt' \frac{\rho_{st}(s', t')}{(s' - s)(t' - t)}
  \]

  Integration ranges defined by the support of the double spectral densities $\rho$

- **Boundaries of $\rho$ are given lowest lying intermediate states**

  \[(I) (II) (III) (IV)\]

  They limit the range of validity of the HDRS:

  - **Pw expansion converge**
    $\Rightarrow z = \cos \theta \in \text{Lehmann ellipses} \ [\text{Lehmann (1958)}]$  

  - The hyperbolae $(s - a)(u - a) = b$ does not enter any double spectral region
    $\Rightarrow$ for a value of $a$, constraints on $b$ yield ranges in $s$ & $t$
Dispersion relation for the scalar form factor of the nucleon

- **Unitarity relation:** \( \text{Im} \, \sigma(t) = \frac{2}{4m^2-t} \left\{ \frac{3}{4} \sigma^\pi_t \left( F^S_\pi(t) \right)^* f_+^0(t) + \sigma^K_t \left( F^K_\pi(t) \right)^* h_+^0(t) \right\} \)

- **Once subtracted dispersion relation:** \( \sigma(t) = \sigma_{\pi N} + \frac{t}{\pi} \int_{t^\pi}^{\infty} \text{d}t' \frac{\text{Im} \sigma(t')}{t'(t'-t)} \)

\[ \Delta \sigma = \sigma \left( 2M^2_\pi \right) - \sigma_{\pi N} \]
Dispersion relation for the $\pi N$ amplitude

- t-channel expansion of the subtracted pseudo-Born amplitude

\[
\bar{D}(\nu = 0, t) = 4\pi \left\{ \frac{1}{p_t^2} \bar{f}_0^+(t) + \frac{5}{2} q_i^2 \bar{f}_2^+(t) + \frac{27}{8} p_t^2 q_i^4 \bar{f}_4^+(t) + \frac{56}{16} p_t^4 q_i^6 \bar{f}_6^+(t) + \cdots \right\}
\]

- Insert $t$-channel RS equations for Born-term-subtracted amplitudes $\bar{f}_J^+(t)$

\[
\bar{D}(\nu = 0, t) = d_{00}^+ + d_{01}^+ t - 16t^2 \int_{t_\pi}^{\infty} dt' \frac{\text{Im} \bar{f}_0^+(t')}{t'^2(t' - 4m^2)(t' - t)} + \{J \geq 2\} + \{s\text{-channel integral}\}
\]

- $\Delta_D = F_{\pi}^2 \left( \bar{D}(\nu = 0, t) - d_{00}^+ + d_{01}^+ t \right)$ from evaluation at $t = 2M_{\pi}^2$
Summary: $\sigma$-term corrections

- **Nucleon scalar form factor**
  \[
  \Delta_\sigma = (13.9 \pm 0.3) \text{ MeV} \\
  + Z_1 \left( \frac{g^2}{4\pi} - 14.28 \right) + Z_2 \left( d^+_{00} M_\pi + 1.46 \right) + Z_3 \left( d^+_{01} M^3_\pi - 1.14 \right) + Z_4 \left( b^+_{00} M^3_\pi + 3.54 \right)
  \]
  \[
  Z_1 = 0.36 \text{ MeV} , \quad Z_2 = 0.57 \text{ MeV} , \quad Z_3 = 12.0 \text{ MeV} , \quad Z_4 = -0.81 \text{ MeV}
  \]

- **$\pi N$ amplitude**
  \[
  \Delta_D = (12.1 \pm 0.3) \text{ MeV} \\
  + \hat{Z}_1 \left( \frac{g^2}{4\pi} - 14.28 \right) + \hat{Z}_2 \left( d^+_{00} M_\pi + 1.46 \right) + \hat{Z}_3 \left( d^+_{01} M^3_\pi - 1.14 \right) + \hat{Z}_4 \left( b^+_{00} M^3_\pi + 3.54 \right)
  \]
  \[
  \hat{Z}_1 = 0.42 \text{ MeV} , \quad \hat{Z}_2 = 0.67 \text{ MeV} , \quad \hat{Z}_3 = 12.0 \text{ MeV} , \quad \hat{Z}_4 = -0.77 \text{ MeV}
  \]

$\leftrightarrow$ most of the dependence on the $\pi N$ parameters cancels in the difference

**Full Correction**

\[
\Delta_D - \Delta_\sigma = (-1.8 \pm 0.2) \text{ MeV}
\]
Hadronic atoms: constraints for $\pi N$

$$\tilde{a}^+ = a^+ + \frac{1}{1+M_\pi/m_p} \left\{ \frac{M_\pi^2 - m_\pi^2}{\pi F_\pi^2} c_1 - 2\alpha f_1 \right\}$$

- $\pi H/\pi D$: bound state of $\pi^-$ and p/d, spectrum sensitive to threshold $\pi N$ amplitude
- Combined analysis of $\pi H$ and $\pi D$:
  - $a_0^+ \equiv a^+ = (7.5 \pm 3.1) \cdot 10^{-3} M_\pi^{-1}$
  - $a_0^- \equiv a^- = (86.0 \pm 0.9) \cdot 10^{-3} M_\pi^{-1}$
- Large $a^+$ suggests a large $\sigma_{\pi N}$,

$$\frac{a_{\pi^-} - p + a_{\pi^+} + p}{2} = (-1.1 \pm 0.9) \cdot 10^{-3} M_\pi^{-1}$$

- Isospin breaking in $\sigma_{\pi N}$ could be important
- Revisit the Cheng-Dashen low-energy theorem

[Baru et al. 2011]
Goldberger-Miyazawa-Oehme sum rule

- Fixed-\(t\) dispersion relations at threshold \(\Leftrightarrow\) GMO sum rule

\[
\frac{g^2}{4\pi} = \left( \left( \frac{m_p + m_n}{M_\pi} \right)^2 - 1 \right) \left\{ \left( 1 + \frac{M_\pi}{m_p} \right) \frac{M_\pi}{4} \left( a_{\pi-p} - a_{\pi+p} \right) - \frac{M_\pi^2}{2} J^- \right\}
\]

\[
J^- = \frac{1}{4\pi^2} \int_0^\infty \frac{dk \sigma_{\pi-p}^{\text{tot}}(k) - \sigma_{\pi+p}^{\text{tot}}(k)}{\sqrt{M_\pi^2 + k^2}}
\]

- \(J^-\) known very accurately Ericson et al. 2002, Abaev et al. 2007
- other determinations

<table>
<thead>
<tr>
<th>de Swart et al. 97</th>
<th>Arndt et al. 94</th>
<th>Ericson et al. 02</th>
<th>Bugg et al. 73</th>
<th>KH80</th>
</tr>
</thead>
<tbody>
<tr>
<td>method</td>
<td>NN</td>
<td>(\pi N)</td>
<td>GM0</td>
<td>(\pi N)</td>
</tr>
<tr>
<td>(g^2/4\pi)</td>
<td>13.54 ± 0.05</td>
<td>13.75 ± 0.15</td>
<td>14.11 ± 0.20</td>
<td>14.30 ± 0.18</td>
</tr>
<tr>
<td>13.76 ± 0.12 ± 0.15</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

- With KH80 scattering lengths \(g^2/4\pi = 14.28\) MeV is reproduced exactly \(\Leftrightarrow\) discrepancy related to old scattering length values
Define as **isoscalar** as

\[
X^+ \rightarrow X^p = \frac{1}{2} \left( X_{\pi^+ p \rightarrow \pi^+ p} + X_{\pi^- p \rightarrow \pi^- p} \right), \quad X \in \{ D, d_{00}, d_{01}, \ldots \}
\]

and "**isospin limit**" by proton and charged pion

- Assume virtual photons to be removed
  - scenario closest to actual \( \pi N \) PWA
- Calculate **IV corrections** in SU(2) ChPT, mainly due to \( \Delta_\pi = M^2_\pi - M^2_{\pi^0} \)
Spin-independent WIMP–nucleon scattering

- Effective Lagrangian
  \[ \mathcal{L} = C_{qq}^{SS} \frac{m_q}{\Lambda^3} \bar{\chi} \chi \bar{q} q + C_{qq}^{VV} \frac{m_q}{\Lambda^2} \bar{\chi} \gamma^\mu \chi \bar{q} \gamma_\mu q + \tilde{C}_{gg}^{S} \frac{\alpha_S}{\Lambda^3} \bar{\chi} \chi G_{\mu\nu} G^{\mu\nu} \]

- WIMP \( \chi \) Dirac fermion and SM singlet
- Spin–independent cross section at vanishing momentum transfer
  \[ \sigma_{SI}^N = \frac{\mu_\chi^2}{\Lambda^4} \left| \left( \frac{m_N}{\Lambda} C_{qq}^{SS} f_q^N - 12 \pi C_{gg}^{S} f_Q^N \right) + C_{qq}^{VV} f_V^N \right|^2 \]
  \[ \mu_\chi = \frac{m_\chi m_N}{m_\chi + m_N}, \quad f_q^N = 2, \quad f_q^N = \frac{\sigma_{\pi N}(1 - \xi)}{m_N} + \Delta f_q^N \]

- Nucleon-matrix elements dominated by \( \sigma_{\pi N} \)